DATA

LARGE AMPLITUDE VIBRATION OF BUCKLED BEAMS AND RECTANGULAR PLATES

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Nomenclature

a, b, h	=	plate width, length, and thickness, (x, y, z directions) respectively
r	=	a/b, plate aspect ratio
t	=	time
u, v, w	=	displacements in the x, y, z directions, respectively
D	=	plate flexural rigidity, $Eh^3/12(1-\nu^2)$
EI	=	beam flexural rigidity
F	=	stress function
ρ	=	mass density
ν	=	Poisson's ratio

Other symbols are defined in the text.

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Introduction

In recent years a number of investigations of the large amplitude vibration of beams[1-4] and flat rectangular plates[5-8] have been reported in which the ends of the beams and the edges of the plates have been assumed to remain a fixed distance apart during vibration. In particular Burgreen[2] has considered the free vibration of a simply supported beam which has been given an initial end displacement and the author[8] has considered free and forced vibration of simply supported and clamped beams and rectangular plates for which initial end and edge displacements have been prescribed. In both reports a one degree of freedom representation of the equations of motion is used. Results are obtained for edge displacements in the postbuckling as well as the prebuckling region. In the case of forced motion, however, the results were restricted to symmetrical motion about the flat position of the beam or plate. For the buckled beam or plate it is also possible to have vibration about the static buckled position. This has been discussed for free vibration in the above reports and it is the purpose of the following remarks to extend the discussion to a case of forced motion.

Equations of Motion

The differential equation of motion for a beam of unit width is

$$\rho h w,_{tt} + (EI w,_{yy}),_{yy} - \frac{Eh}{b}\left[v_o + \frac{1}{2}\int_o^b (w,_y)^2 dy \right] w,_{yy} = P(y,t) \qquad (1)$$

[redacted]

where v_o represents an initial axial displacement measured from the unstressed state. For a plate the dynamic von Kármán equations are

$$\nabla^4 F = E\,(w_{,xy}^2 - w_{,xx}\,w_{,yy})$$

$$D\nabla^4 w - h\,(F_{,yy}\,w_{,xx} + F_{,xx}\,w_{,yy} - 2F_{,xy}\,w_{,xy}) + \rho h\,w_{,tt} = P(x,y,t) \tag{2}$$

where

$$\sigma_x = F_{,yy}\ , \quad \sigma_y = F_{,xx}\ , \quad \tau_{xy} = -F_{,xy}$$

are the membrane stresses. When a single mode is assumed and Galerkin's method is applied, the problem reduces to the solution of a single ordinary differential equation in time.

In the case of a simply supported beam, for example, we assume

$$w(y,t) = b\,\xi(t)\,\sin\frac{\pi y}{b} \tag{3}$$

and obtain the following equation in nondimensional form

$$\xi_{,\tau\tau} + p\,\xi + q\,\xi^3 = f(\tau) \tag{4}$$

where

$$p = \frac{\pi^4}{12}\,\gamma^2\,(1-\lambda) \qquad q = \frac{\pi^4}{4} \qquad \tau = \left(\frac{E}{\rho}\right)^{\frac{1}{2}}\frac{t}{b} \qquad \gamma = \frac{h}{b}$$

The parameter λ is a measure of the initial axial displacement and is defined

$$\lambda = \frac{v_o}{v_{o_{cr}}} \tag{5}$$

where $v_{o_{cr}}$ is the axial displacement which produces the buckling load. Thus $\lambda > 1$ refers to the postbuckling region. An equation of the same form is obtained for other beam boundary conditions and for plates as well. The coefficients p and q for simply supported and clamped beams and rectangular plates are given in Ref. 8. The remarks which follow apply to these cases as well as others which may be defined.

To study the motion about the static buckled position it is convenient to change to the variable

$$\delta = \xi - \xi_1 \tag{6}$$

where ξ_1 is the static buckle amplitude and δ is the variation from that position. If Eq. (6) is substituted into Eq. (4) it follows that for harmonic forcing

$$\delta,_{\tau\tau} + \omega_o^2 \delta + c_2 \delta^2 + c_3 \delta^3 = \overline{f} \cos \omega \tau \tag{7}$$

where

$$\omega_o^2 = p + 3q\,\xi_1^2$$

$$c_2 = 3q\,\xi_1$$

$$c_3 = q$$

Note that ω_o is the linear vibration frequency about the buckled position. The problem of small amplitude vibration of a buckled plate has been more fully discussed elsewhere.[9-10]

This equation is of similar form to an equation derived for the vibration of initially curved plates and shells.[11-13] The Linstedt-Duffing perturbation technique[11, 13] used in two of the above reports may be applied here. Let

$$
\begin{aligned}
\delta &= \delta_o + \alpha\,\delta_1 + \alpha^2\,\delta_2 + \alpha^3\,\delta_3 + \ldots \\
\omega^2 &= \omega_o^2 + \alpha\,\omega_1^2 + \alpha^2\,\omega_2^2 + \alpha^3\,\omega_3^2 + \ldots \\
\overline{f} &= \overline{f}_o + \alpha\,\overline{f}_1 + \alpha^2\,\overline{f}_2 + \alpha^3\,\overline{f}_3 + \ldots
\end{aligned}
\tag{8}
$$

where the initial conditions on Eq. (7) are taken to be

$$\delta(0) = \alpha \qquad \delta_{,\tau}(0) = 0 \tag{9}$$

from which it follows

$$\delta_1(0) = 1 \quad , \qquad \delta_o(0) = \delta_2(0) = \delta_3(0) = \ldots = 0 \tag{10}$$

It is convenient to introduce the forcing function as follows,

let
$$\overline{f}_o = \overline{f}_1 = \overline{f}_2 = 0 \tag{11}$$

so that
$$\overline{f} = \alpha^3\,\overline{f}_3 \tag{12}$$

Then it follows that when Eqs. (8) are substituted into Eq. (7) and terms are collected according to the power of α a series of equations are obtained. The first is

$$\alpha^o : \qquad \delta_{o,\tau\tau} + \omega^2\,\delta_o + c_2\,\delta_o^2 + c_3\,\delta_o^3 = 0 \tag{13}$$

which has as its solution, in view of Eqs. (10)

$$\delta_o(\tau) = 0 \tag{14}$$

and next

$$\alpha^1 : \qquad \delta_{1,\tau\tau} + \omega^2\,\delta_1 = 0 \tag{15}$$

which has as its solution

$$\delta_1(\tau) = \cos\omega\tau \tag{16}$$

Continuing, we obtain

$$\alpha^2 : \qquad \delta_{2,\tau\tau} + \omega^2\,\delta_2 = -\frac{c_2}{2} + \omega_1^2\cos\omega\tau - \frac{c_2}{2}\cos 2\,\omega\tau \tag{17}$$

To insure a periodic solution it is necessary that

$$\omega_1^2 = 0 \tag{18}$$

thus the solution to Eq. (17) becomes

$$\delta_2(\tau) = \frac{c_2}{6\,\omega^2}\,(-3 + 2\cos\omega\tau + \cos 2\omega\tau) \tag{19}$$

Finally

$$\alpha^3 : \qquad \delta_{3,\tau\tau} + \omega^2\delta_3 = -\frac{c_2^2}{3\,\omega^2} + \left(\omega_2^2 + \frac{5}{6}\,\frac{c_2^2}{\omega^2} - \frac{3}{4}\,c_3 + \bar{f}_3\right)\cos\omega\tau$$

$$-\frac{c_2^2}{3\,\omega^2}\cos 2\omega\tau - \left(\frac{c_2^2}{3\,\omega^2} + \frac{c_3}{4}\right)\cos 3\omega\tau \tag{20}$$

Once again, to insure a periodic solution, it is necessary that

$$\omega_2^2 = -\frac{5}{6}\,\frac{c_2^2}{\omega^2} + \frac{3\,c_3}{4} - \bar{f}_3 \tag{21}$$

which, from Eqs. (8) and (12), may be written

$$\omega^2 = \omega_o^2 + \alpha^2\left(\frac{3\,c_3}{4} - \frac{5}{6}\,\frac{c_2^2}{\omega^2} - \frac{\bar{f}}{\alpha}\right) \tag{22}$$

It is worth noting that the Ritz-Galerkin and related methods as they are commonly applied are inadequate for obtaining an approximate solution to Eq. (7). It is common practice to use the solution of the corresponding linear equation as an assumed solution of the non-linear equation. The frequency-amplitude relation is obtained by means of a certain time integration over a cycle of the motion which minimizes the error introduced by this assumption. Unfortunately the restriction imposed by the assumed solution is such that all contributions of the term $c_2\delta^2$ are lost in the integration regardless of its actual influence. If, however, more care is used in the selection of an assumed function, this difficulty may be overcome.

In our case let

$$\delta(\tau) = A + B \cos \omega\tau + C \cos 2\omega\tau \tag{23}$$

and preceed with the Ritz-Galerkin method as described, for example, in Ref. 14. We then obtain four algebraic equation in the four unknowns ω^2, A, B, and C but because of the complexity of the equations no general algebraic solution is possible. If, in addition, we let

$$\begin{aligned}
\omega^2 &= \omega_o^2 + \alpha\, \omega_1^2 + \alpha^2 \omega_2^2 + \dots \\
A &= \alpha A_1 + \alpha^2 A_2 + \alpha^3 A_3 + \dots \\
B &= \alpha B_1 + \alpha^2 B_2 + \alpha^3 B_3 + \dots \\
C &= \alpha C_1 + \alpha^2 C_2 + \alpha^3 C_3 + \dots
\end{aligned} \tag{24}$$

and solve the resulting equations we arrive at the identical frequency-amplitude relation given by Eq. (22).*

Free Vibration

The relation for free vibration may be obtained by setting $\overline{f} = 0$ in Eq. (22). An exact solution for free vibration is also possible in this case in terms of elliptic functions. It is

$$\delta(\tau) = (\alpha + \mu)\, \mathrm{dn}\,(\overline{\omega}\tau, k) - \mu \tag{25}$$

where

$$\mu = \frac{c_2}{3 c_3} \qquad \overline{\omega}^2 = \frac{c_3 (\alpha + \mu)^2}{2} \qquad k^2 = \frac{(\omega_o^2 - c_2 \mu)}{\overline{\omega}^2} + 2$$

The initial conditions are

$$\delta(0) = \alpha \qquad \delta_{,\tau}(0) = 0 \tag{26}$$

where α is positive and subject to the condition

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*This was pointed out to me by Professor E. F. Masur.

$$-2 < \frac{(\omega_o^2 - c_2\mu)}{c_3(\alpha+\mu)^2} < -1 \tag{27}$$

Physically this restricts consideration to motion about the buckled position on one side of the flat position. The period of the motion is given by the elliptic integral

$$T = \frac{2K}{\overline{\omega}} = \frac{2}{\overline{\omega}} \int_0^{\pi/2} \frac{d\phi}{1 - k^2 \sin^2\phi} \tag{28}$$

It should be noted that Eq. (25) is not a general solution to Eq. (7) for arbitrary values of the coefficients ω_o^2, c_2, and c_3 but only for free motion when the relation

$$9\,\omega_o^2 c_3 - 2 c_2^2 = 0 \tag{29}$$

holds. This condition is satisfied in this case because Eq. (7) was obtained from Eq. (4). An exact solution in terms of elliptic functions is still possible, however, as described in Ref. 12, for example.

Numerical Results

Numerical results have been obtained from Eqs. (22) and (28) for the special case of a buckled beam with $\lambda = 2$ and $\gamma = 0.005$ and are presented in Fig. 1 for free motion. In this figure the amplitude is given in terms of the number of beam thicknesses and the frequency in terms of the square of the ratio of the nonlinear to the linear frequency. Since for the exact solution the motion is not symmetrical about the undeflected position the curve for a negative initial condition differs from that for a positive initial condition. A typical deflection curve for a cycle of the motion is shown in the lower right of the figure. The negative amplitude α' is related to α by

$$\alpha' = (\alpha + \mu)(1 - k^2)^{\frac{1}{2}} - \mu \qquad (30)$$

In Fig. 2 the dynamic response of the same beam to harmonic forcing is shown as obtained from Eq. (22).

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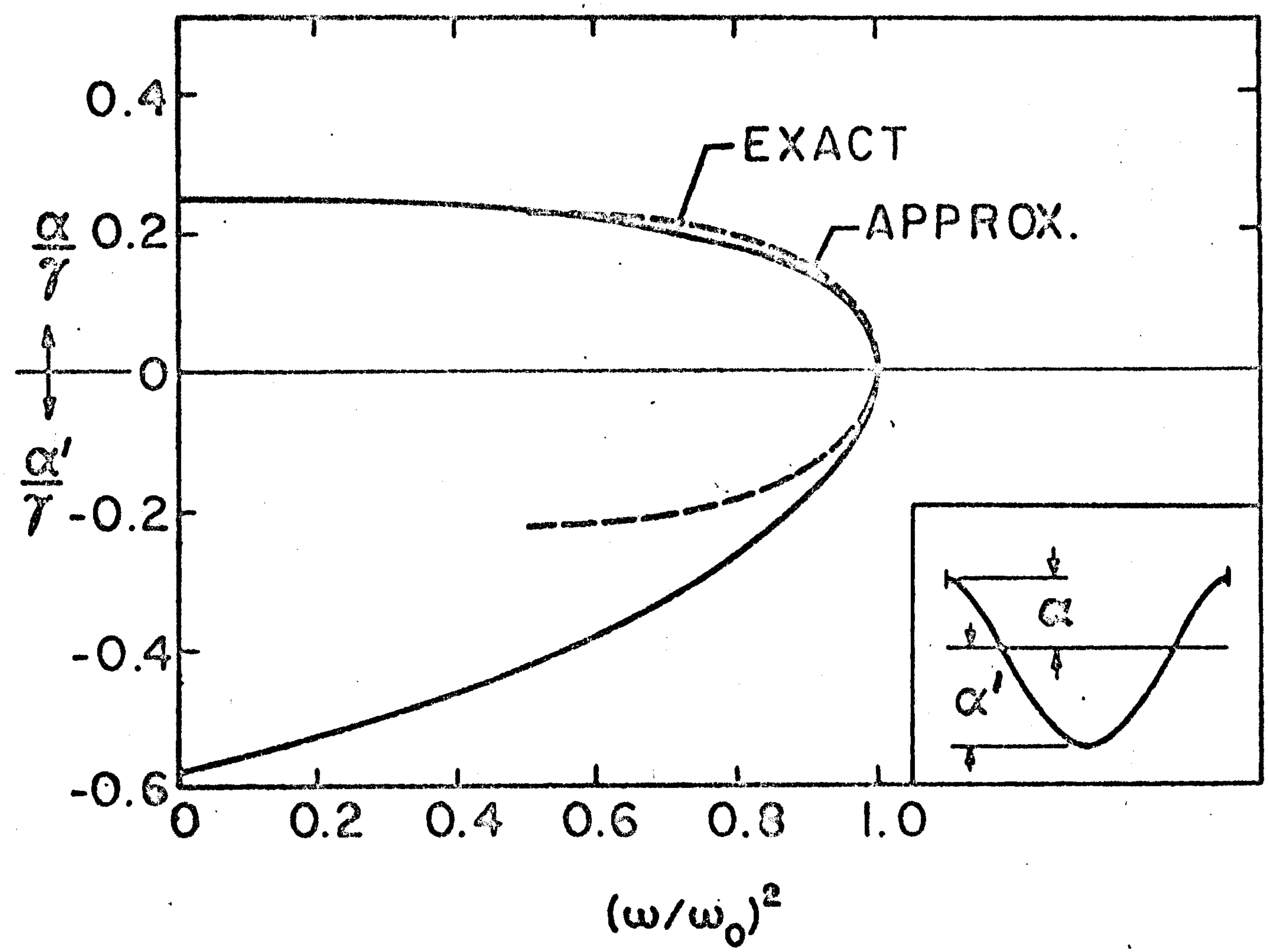
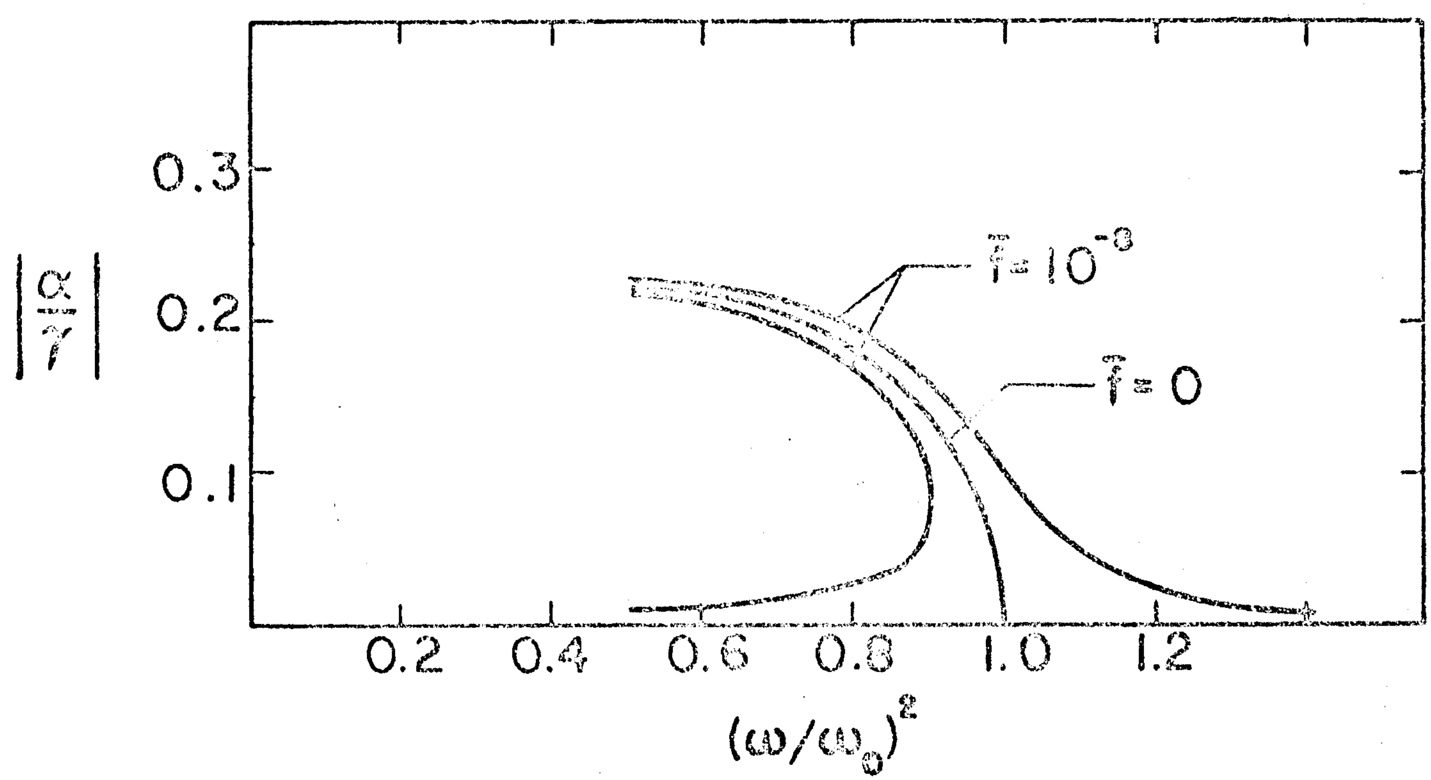
Figure 1. Free vibration.

Figure 2. Forced vibration.



EXACT
APPROX.
$\frac{\alpha}{\gamma}$
$\frac{\alpha'}{\gamma}$
0.4
0.2
0
-0.2
-0.4
-0.6
0
0.2
0.4
0.6
0.8
1.0
$(\omega/\omega_0)^2$
α
α'



$\left|\frac{\alpha}{\gamma}\right|$
0.3
0.2
0.1
$\tilde{f} = 10^{-3}$
$\tilde{f} = 0$
0.2
0.4
0.6
0.8
1.0
1.2
$(\omega/\omega_0)^2$